Solution to Exercise 12.23

⋆12.23 Consider the expression $(\boldsymbol{\Gamma}^\top \otimes \mathbf{I}_n)\boldsymbol{y}_\bullet$, in the notation of Section 12.5. Show that it is equal to a gn–vector that can be written as

$$\begin{bmatrix} \boldsymbol{Y}\boldsymbol{\gamma}_1 \\ \vdots \\ \boldsymbol{Y}\boldsymbol{\gamma}_m \end{bmatrix},$$

where $\boldsymbol{\gamma}_i$, $i = 1, \ldots, g$, is the i^{th} column of $\boldsymbol{\Gamma}$.

Show similarly that $(\boldsymbol{\Gamma}^\top \otimes \mathbf{I}_n)(\mathbf{I}_g \otimes \boldsymbol{WB})\boldsymbol{\gamma}^\bullet$ is equal to a gn–vector that can be written as

$$\begin{bmatrix} \boldsymbol{W}\boldsymbol{b}_1 \\ \vdots \\ \boldsymbol{W}\boldsymbol{b}_m \end{bmatrix},$$

where $\boldsymbol{b}_i$ is the i^{th} column of $\boldsymbol{B}$.

Using these results, demonstrate that $(\boldsymbol{\Gamma}^\top \otimes \mathbf{I}_n)\big(\boldsymbol{y}_\bullet - (\mathbf{I}_g \otimes \boldsymbol{WB})\boldsymbol{\gamma}^\bullet\big)$ is equal to $\boldsymbol{y}_\bullet - \boldsymbol{X}_\bullet\boldsymbol{\beta}_\bullet$. Explain why this proves the result (12.108).

The matrix $\boldsymbol{\Gamma}^\top \otimes \mathbf{I}_n$ is the $gn \times gn$ matrix

$$\begin{bmatrix} \gamma_{11}\mathbf{I}_n & \gamma_{21}\mathbf{I}_n & \cdots & \gamma_{g1}\mathbf{I}_n \\ \gamma_{12}\mathbf{I}_n & \gamma_{22}\mathbf{I}_n & \cdots & \gamma_{g2}\mathbf{I}_n \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{1g}\mathbf{I}_n & \gamma_{2g}\mathbf{I}_n & \cdots & \gamma_{gg}\mathbf{I}_n \end{bmatrix}.$$

Postmultiplying it by $\boldsymbol{y}_\bullet$ yields the gn–vector

$$\begin{bmatrix} \gamma_{11}\boldsymbol{y}_1 + \gamma_{21}\boldsymbol{y}_2 + \cdots + \gamma_{g1}\boldsymbol{y}_g \\ \gamma_{12}\boldsymbol{y}_1 + \gamma_{22}\boldsymbol{y}_2 + \cdots + \gamma_{g2}\boldsymbol{y}_g \\ \vdots \\ \gamma_{1g}\boldsymbol{y}_1 + \gamma_{2g}\boldsymbol{y}_2 + \cdots + \gamma_{gg}\boldsymbol{y}_g \end{bmatrix}.$$

Each subvector here is what we get if we postmultiply the matrix $\boldsymbol{Y}$ by one of the columns of $\boldsymbol{\Gamma}$. Thus this vector can also be written as

$$\begin{bmatrix} \boldsymbol{Y}\boldsymbol{\gamma}_1 \\ \vdots \\ \boldsymbol{Y}\boldsymbol{\gamma}_m \end{bmatrix},$$

as we were required to show.

The matrix $(\boldsymbol{\Gamma}^\top \otimes \mathbf{I}_n)(\mathbf{I}_g \otimes \boldsymbol{WB}) = \boldsymbol{\Gamma}^\top \otimes \boldsymbol{WB}$ is

$$\begin{bmatrix} \gamma_{11}\boldsymbol{WB} & \gamma_{21}\boldsymbol{WB} & \cdots & \gamma_{g1}\boldsymbol{WB} \\ \gamma_{12}\boldsymbol{WB} & \gamma_{22}\boldsymbol{WB} & \cdots & \gamma_{g2}\boldsymbol{WB} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{1g}\boldsymbol{WB} & \gamma_{2g}\boldsymbol{WB} & \cdots & \gamma_{gg}\boldsymbol{WB} \end{bmatrix}, \tag{S12.32}$$

and the vector $\boldsymbol{\gamma}^\bullet$ is

$$\begin{bmatrix} \boldsymbol{\gamma}^1 \\ \boldsymbol{\gamma}^2 \\ \vdots \\ \boldsymbol{\gamma}^g \end{bmatrix}. \tag{S12.33}$$

If we postmultiply (S12.32) by (S12.33), we obtain the gn–vector

$$\begin{bmatrix} \gamma_{11}\boldsymbol{WB\gamma}^1 + \gamma_{21}\boldsymbol{WB\gamma}^2 + \cdots + \gamma_{g1}\boldsymbol{WB\gamma}^g \\ \gamma_{12}\boldsymbol{WB\gamma}^1 + \gamma_{22}\boldsymbol{WB\gamma}^2 + \cdots + \gamma_{g2}\boldsymbol{WB\gamma}^g \\ \vdots \\ \gamma_{1g}\boldsymbol{WB\gamma}^1 + \gamma_{2g}\boldsymbol{WB\gamma}^2 + \cdots + \gamma_{gg}\boldsymbol{WB\gamma}^g \end{bmatrix}. \tag{S12.34}$$

Now, for $i = 1, \ldots, g$,

$$\begin{aligned} \gamma_{1i}\boldsymbol{WB\gamma}^1 + \gamma_{2i}\boldsymbol{WB\gamma}^2 + \cdots + \gamma_{gi}\boldsymbol{WB\gamma}^g &= \boldsymbol{WB}\sum_{j=1}^{g} \boldsymbol{\gamma}^j \gamma_{ji} \\ &= \boldsymbol{WBe}_i = \boldsymbol{Wb}_i. \end{aligned}$$

Here we have used the fact that, because $\boldsymbol{\Gamma}^{-1}\boldsymbol{\Gamma} = \mathbf{I}_g$, $\sum \boldsymbol{\gamma}^j \gamma_{ji} = \boldsymbol{e}_i$, the i^{th} unit basis vector in $\mathbb{R}^g$. The last equality follows from the fact that $\boldsymbol{Be}_i$ is just $\boldsymbol{b}_i$, the i^{th} column of $\boldsymbol{B}$.

From these two results, we can see that $(\boldsymbol{\Gamma}^\top \otimes \mathbf{I}_n)\big(\boldsymbol{y}_\bullet - (\mathbf{I}_g \otimes \boldsymbol{WB})\boldsymbol{\gamma}^\bullet\big)$ is the vector formed by vertically stacking the columns of the $n \times g$ matrix

$$\boldsymbol{Y\Gamma} - \boldsymbol{WB}. \tag{S12.35}$$

But, from (12.55), the vector $\boldsymbol{y}_\bullet - \boldsymbol{X}_\bullet\boldsymbol{\beta}_\bullet$ is also the vector formed by stacking the columns of (S12.35). Since, by (12.106),

$$\boldsymbol{y}_\bullet - \boldsymbol{W}_\bullet\boldsymbol{\pi}_\bullet = \boldsymbol{y}_\bullet - (\mathbf{I}_g \otimes \boldsymbol{WB})\boldsymbol{\gamma}^\bullet,$$

we conclude that

$$(\boldsymbol{\Gamma}^\top \otimes \mathbf{I}_n)(\boldsymbol{y}_\bullet - \boldsymbol{W}_\bullet\boldsymbol{\pi}_\bullet) = \boldsymbol{y}_\bullet - \boldsymbol{X}_\bullet\boldsymbol{\beta}_\bullet,$$

which is equation (12.108).