Solution to Exercise 12.2

*12.2 Prove the first result of equations (12.08) for an arbitrary $p \times q$ matrix $A$ and an arbitrary $r \times s$ matrix $B$. Prove the second result for $A$ and $B$ as above, and for $C$ and $D$ arbitrary $q \times t$ and $s \times u$ matrices, respectively. Prove the third result in (12.08) for an arbitrary nonsingular $p \times p$ matrix $A$ and nonsingular $r \times r$ matrix $B$.

Give details of the interchanges of rows and columns needed to convert $A \otimes B$ into $B \otimes A$, where $A$ is $p \times q$ and $B$ is $r \times s$.

The first result to be shown is that

$$(A \otimes B)^\top = A^\top \otimes B^\top,$$

where $A$ is $p \times q$, and $B$ is $r \times s$. Each element of $A \otimes B$ is of the form $a_{ij} b_{kl}$, and this particular element is in row $(i - 1)r + k$ of the Kronecker product, and in column $(j - 1)s + l$. This element is thus the element of $(A \otimes B)^\top$ in row $(j - 1)s + 1$ and column $(i - 1)r + k$. Now $a_{ij} = A_{ji}^\top$ and $b_{kl} = B_{lk}^\top$. Thus $a_{ij} b_{kl}$ is in row $(j - 1)s + l$ and column $(i - 1)r + k$ of $A^\top \otimes B^\top$. This shows that $(A \otimes B)^\top$ and $A^\top \otimes B^\top$ coincide element by element, and therefore as complete matrices.

The second result to be shown is that

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD), \quad (S12.03)$$

for $A$ and $B$ as above, and $C$ and $D$, respectively, $q \times t$ and $s \times u$. We have that $A \otimes B$ is $pr \times qs$, $C \otimes D$ is $qs \times tu$, $AC$ exists and is $p \times t$, and $BD$ exists and is $r \times u$. Thus $(A \otimes B)(C \otimes D)$ exists and is $pr \times tu$, while $(AC) \otimes (BD)$ is $pr \times tu$. Thus all the dimensions are correct.

Write out the left-hand side of equation (S12.03) explicitly as

$$
\begin{bmatrix}
  a_{11} B & \cdots & a_{1q} B \\
  \vdots & \ddots & \vdots \\
  a_{p1} B & \cdots & a_{pq} B \\
\end{bmatrix}
\begin{bmatrix}
  c_{11} D & \cdots & c_{1t} D \\
  \vdots & \ddots & \vdots \\
  c_{q1} D & \cdots & c_{qt} D \\
\end{bmatrix}.
$$

The partitioning of both factors is compatible for multiplication, and we see directly that the product can be written as

$$
\begin{bmatrix}
  \sum_{j=1}^q a_{1j} c_{j1} BD & \cdots & \sum_{j=1}^q a_{1j} c_{jt} BD \\
  \vdots & \ddots & \vdots \\
  \sum_{j=1}^q a_{pj} c_{j1} BD & \cdots & \sum_{j=1}^q a_{pj} c_{jt} BD \\
\end{bmatrix}, \quad (S12.04)
$$

But $\sum_{j=1}^q a_{ij} c_{jk}$ is element $(i, k)$ of $AC$, and so it is clear that the right-hand side of equation (S12.04) is just $(AC) \otimes (BD)$. 

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The third result to be shown is that

\[(A \otimes B)^{-1} = A^{-1} \otimes B^{-1},\]

where \(A\) is \(p \times p\) and \(B\) is \(r \times r\), both nonsingular. Form the \(pr \times pr\) product \((A \otimes B)(A^{-1} \otimes B^{-1})\). By the previous result, this product is

\[AA^{-1} \otimes BB^{-1} = I_p \otimes I_r.\]

If the last expression is written out explicitly, we obtain the partitioned matrix

\[
\begin{bmatrix}
I_r & O & \cdots & O \\
O & I_r & \cdots & O \\
\vdots & \vdots & \ddots & \vdots \\
O & O & \cdots & I_r
\end{bmatrix},
\]

with \(p\) rows and columns of blocks. This matrix is manifestly just \(I_{pr}\), which shows that \(A^{-1} \otimes B^{-1}\) is the inverse of \(A \otimes B\).

Finally, we can answer the last part of the question. Since \(a_{ij} b_{kl}\) is both element \(((i-1)r+k, (j-1)s+l)\) of \(A \otimes B\) and element \(((k-1)p+i, (l-1)q+j)\) of \(B \otimes A\), in order to go from \(A \otimes B\) to \(B \otimes A\), we must move row \((i-1)r+k\) of the former to row \((k-1)p+i\) for all \(i\) and \(k\) within the dimensions, and then move column \((j-1)s+l\) of the former to column \((l-1)q+j\) for all \(j\) and \(l\) within the dimensions. This prescription is unique, since, as \(i\) and \(k\) vary, \((i-1)r+k\) varies with no repetitions from 1 to \(pr\), as does \((k-1)p+i\), with a similar result for the columns. This implies that we have defined a unique permutation of both rows and columns.