Solution to Exercise 12.1

$\star$**12.1** Show that the $gn \times gn$ covariance matrix $\boldsymbol{\Sigma}_\bullet$ defined in equation (12.07) is positive definite if and only if the $g \times g$ matrix $\boldsymbol{\Sigma}$ used to define it is positive definite.

Partition the $gn \times 1$ vector $\boldsymbol{x}_\bullet$ as $[\boldsymbol{x}_1 \vdots \cdots \vdots \boldsymbol{x}_g]$. Then the quadratic form $\boldsymbol{x}_\bullet^\top \boldsymbol{\Sigma}_\bullet \boldsymbol{x}_\bullet$ can be written as

$$\boldsymbol{x}_\bullet^\top \boldsymbol{\Sigma}_\bullet \boldsymbol{x}_\bullet = \sum_{i=1}^{g} \sum_{j=1}^{g} \sigma_{ij} \boldsymbol{x}_i^\top \boldsymbol{x}_j = \mathrm{Tr}(\boldsymbol{\Sigma} \boldsymbol{X}^\top \boldsymbol{X}), \tag{S12.01}$$

where the $n \times g$ matrix $\boldsymbol{X} \equiv [\boldsymbol{x}_1 \;\cdots\; \boldsymbol{x}_g]$.

Choose an arbitrary nonzero g–vector $\boldsymbol{z}$, and construct the gn–vector $\boldsymbol{x}_\bullet$ as $\boldsymbol{z} \otimes \boldsymbol{e}_1$, where $\boldsymbol{e}_1$ is the first unit basis vector in E^g. Thus $\boldsymbol{x}_i = z_i \boldsymbol{e}_1$. Using the middle expression in (S12.01), and noting that $\boldsymbol{e}_1^\top \boldsymbol{e}_1 = 1$, we see that

$$\boldsymbol{x}_\bullet^\top \boldsymbol{\Sigma}_\bullet \boldsymbol{x}_\bullet = \sum_{i=1}^{g} \sum_{j=1}^{g} \sigma_{ij} z_i z_j = \boldsymbol{z}^\top \boldsymbol{\Sigma} \boldsymbol{z}.$$

If $\boldsymbol{\Sigma}_\bullet$ is positive definite, then the leftmost expression above is positive, and so also, therefore, is the rightmost one. Since this holds for arbitrary nonzero $\boldsymbol{z}$, this proves that $\boldsymbol{\Sigma}$ is positive definite.

Now suppose that $\boldsymbol{\Sigma}$ is positive definite. The rightmost expression in (S12.01) is equal to

$$\mathrm{Tr}(\boldsymbol{X} \boldsymbol{\Sigma} \boldsymbol{X}^\top) = \sum_{i=1}^{g} \boldsymbol{e}_i^\top \boldsymbol{X} \boldsymbol{\Sigma} \boldsymbol{X}^\top \boldsymbol{e}_i, \tag{S12.02}$$

since each of the terms in the summation is one of the elements of the principal diagonal of the matrix $\boldsymbol{X} \boldsymbol{\Sigma} \boldsymbol{X}^\top$. The i^{th} term must be nonnegative, since it is a quadratic form in the positive definite matrix $\boldsymbol{\Sigma}$, and it is zero only if $\boldsymbol{e}_i^\top \boldsymbol{X}$, which is the i^{th} row of $\boldsymbol{X}$, is zero. Thus the summation on the right of equation (S12.02) is zero if and only if all the elements of $\boldsymbol{X}$ are zero, that is, if and only if all the elements of $\boldsymbol{x}_\bullet$ are zero. Otherwise, at least one term in the summation is positive, and so $\boldsymbol{x}_\bullet^\top \boldsymbol{\Sigma}_\bullet \boldsymbol{x}_\bullet$ is positive for all nonzero $\boldsymbol{x}_\bullet$, which proves that $\boldsymbol{\Sigma}_\bullet$ is positive definite.