

## Solution to Exercise 11.9

**\*11.9** Plot  $\Upsilon_t(\boldsymbol{\beta})$ , which is defined in equation (11.16), as a function of  $\mathbf{X}_t\boldsymbol{\beta}$  for both the logit and probit models. For the logit model only, prove that  $\Upsilon_t(\boldsymbol{\beta})$  achieves its maximum value when  $\mathbf{X}_t\boldsymbol{\beta} = 0$  and declines monotonically as  $|\mathbf{X}_t\boldsymbol{\beta}|$  increases.

For the logit model, using the fact that  $\Lambda(-x) = 1 - \Lambda(x)$ , we see that

$$\Upsilon_t(\boldsymbol{\beta}) = \frac{\lambda^2(\mathbf{X}_t\boldsymbol{\beta})}{\Lambda(\mathbf{X}_t\boldsymbol{\beta})\Lambda(-\mathbf{X}_t\boldsymbol{\beta})}.$$

Using the relationship between  $\lambda(x)$  and  $\Lambda(x)$  given in equations (11.08), this reduces to

$$\Upsilon_t(\boldsymbol{\beta}) = \frac{\Lambda^2(\mathbf{X}_t\boldsymbol{\beta})\Lambda^2(-\mathbf{X}_t\boldsymbol{\beta})}{\Lambda(\mathbf{X}_t\boldsymbol{\beta})\Lambda(-\mathbf{X}_t\boldsymbol{\beta})} = \Lambda(\mathbf{X}_t\boldsymbol{\beta})\Lambda(-\mathbf{X}_t\boldsymbol{\beta}) = \lambda(\mathbf{X}_t\boldsymbol{\beta}).$$

Since

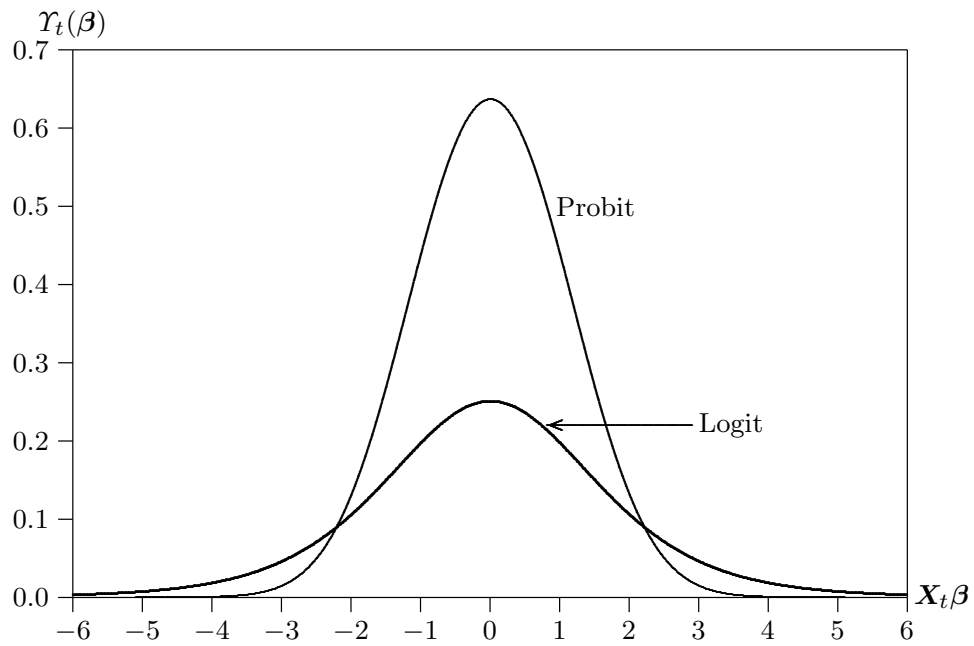
$$\lambda(x) = \frac{e^x}{(1 + e^x)^2},$$

a little algebra shows that the first derivative of  $\lambda(x)$  is

$$\lambda'(x) = \frac{e^x - e^{2x}}{(1 + e^x)^3}. \quad (\text{S11.09})$$

Because  $e^{2x} = e^x$  when  $x = 0$ , we see that the derivative of  $\lambda(x)$  is 0 when  $x = 0$ . Because  $e^{2x} > e^x$  for  $x > 0$  and  $e^{2x} < e^x$  for  $x < 0$ , this derivative must be positive whenever  $x < 0$  and negative whenever  $x > 0$ . Thus, because the denominator of (S11.09) is always positive, it follows that  $\lambda(x)$  must achieve a maximum at  $x = 0$  and decline monotonically as  $|x|$  increases. What is true for  $\lambda(x)$  is also true for  $\Upsilon_t(\boldsymbol{\beta})$  regarded as a function of  $\mathbf{X}_t\boldsymbol{\beta}$ .

Unfortunately, no similarly simple proof appears to be available for the probit model. But the properties of  $\Upsilon_t(\boldsymbol{\beta})$  that hold for the logit model also hold for the probit model, as Figure S11.1 shows. In both cases, the maximum is achieved at  $\mathbf{X}_t\boldsymbol{\beta} = 0$ , and the weights decline monotonically towards zero as  $|\mathbf{X}_t\boldsymbol{\beta}|$  increases.



**Figure S11.1** Weight functions for logit and probit models