Solution to Exercise 11.9

*11.9* Plot $\Upsilon_t(\beta)$, which is defined in equation (11.16), as a function of $X_t\beta$ for both the logit and probit models. For the logit model only, prove that $\Upsilon_t(\beta)$ achieves its maximum value when $X_t\beta = 0$ and declines monotonically as $|X_t\beta|$ increases.

For the logit model, using the fact that $\Lambda(-x) = 1 - \Lambda(x)$, we see that

$$\Upsilon_t(\beta) = \frac{\lambda^2(X_t\beta)}{\Lambda(X_t\beta)\Lambda(-X_t\beta)}.$$  

Using the relationship between $\lambda(x)$ and $\Lambda(x)$ given in equations (11.08), this reduces to

$$\Upsilon_t(\beta) = \frac{\Lambda^2(X_t\beta)\Lambda^2(-X_t\beta)}{\Lambda(X_t\beta)\Lambda(-X_t\beta)} = \Lambda(X_t\beta)\Lambda(-X_t\beta) = \lambda(X_t\beta).$$

Since

$$\lambda(x) = \frac{e^x}{(1 + e^x)^2},$$

a little algebra shows that the first derivative of $\lambda(x)$ is

$$\lambda'(x) = \frac{e^x - e^{2x}}{(1 + e^x)^3}. \quad (S11.09)$$

Because $e^{2x} = e^x$ when $x = 0$, we see that the derivative of $\lambda(x)$ is 0 when $x = 0$. Because $e^{2x} > e^x$ for $x > 0$ and $e^{2x} < e^x$ for $x < 0$, this derivative must be positive whenever $x < 0$ and negative whenever $x > 0$. Thus, because the denominator of (S11.09) is always positive, it follows that $\lambda(x)$ must achieve a maximum at $x = 0$ and decline monotonically as $|x|$ increases. What is true for $\lambda(x)$ is also true for $\Upsilon_t(\beta)$ regarded as a function of $X_t\beta$.

Unfortunately, no similarly simple proof appears to be available for the probit model. But the properties of $\Upsilon_t(\beta)$ that hold for the logit model also hold for the probit model, as Figure S11.1 shows. In both cases, the maximum is achieved at $X_t\beta = 0$, and the weights decline monotonically towards zero as $|X_t\beta|$ increases.
Figure S11.1 Weight functions for logit and probit models