Solution to Exercise 11.32

⋆11.32 Let z be distributed as $\mathrm{N}(0,1)$. Show that $\mathrm{E}(z\,|\,z<x) = -\phi(x)/\Phi(x)$, where Φ and ϕ are, respectively, the CDF and PDF of the standard normal distribution. Then show that $\mathrm{E}(z\,|\,z>x) = \phi(x)/\Phi(-x) = \phi(-x)/\Phi(-x)$. The second result explains why the inverse Mills ratio appears in (11.77).

The density of z conditional on $z<x$ is simply the standard normal density $\phi(z)$ divided by the probability that $z<x$, which is $\Phi(x)$. Thus

$$\mathrm{E}(z\,|\,z<x) = \Phi^{-1}(x)\int_{-\infty}^{x} z\,(2\pi)^{-1/2}\exp(-\tfrac{1}{2}z^2)\,dz.$$

If we let $y = -\frac{1}{2}z^2$, then $dy/dz = -z$, and we can rewrite the right-hand side of this equation as

$$-\Phi^{-1}(x)\int_{-\infty}^{-x^2/2}(2\pi)^{-1/2}\exp(y)\,dy.$$

Since the integral of $\exp(y)$ is simply $\exp(y)$, and $\exp(-\infty)=0$, the definite integral here is just

$$(2\pi)^{-1/2}\exp(-\tfrac{1}{2}x^2) - 0 = \phi(x).$$

Therefore,

$$\mathrm{E}(z\,|\,z<x) = -\frac{\phi(x)}{\Phi(x)},$$

which is the result that was to be proved.

For the second part, the density of z conditional on $z>x$ is $\phi(x)/\Phi(-x)$, where we have used the fact that $\Phi(-x) = 1-\Phi(x)$. Therefore,

$$\mathrm{E}(z\,|\,z>x) = \Phi^{-1}(-x)\int_{x}^{\infty} z\,(2\pi)^{-1/2}\exp(-\tfrac{1}{2}z^2)\,dz.$$

This time, we let $y = \frac{1}{2}z^2$, which implies that $dy/dz = z$. Then we can rewrite the right-hand side as

$$\Phi^{-1}(-x)\int_{x^2/2}^{\infty}(2\pi)^{-1/2}\exp(-y)\,dy.$$

Since the integral of $\exp(-y)$ is $-\exp(-y)$, the definite integral here is

$$0 + (2\pi)^{-1/2}\exp(-\tfrac{1}{2}x^2) = \phi(x).$$

Therefore,

$$\mathrm{E}(z\,|\,z>x) = \frac{\phi(x)}{\Phi(-x)} = \frac{\phi(-x)}{\Phi(-x)},$$

where the second equality follows immediately from the fact that the standard normal density is symmetric around the origin. This is the result that was to be proved.