Solution to Exercise 11.26

⋆11.26 Suppose that y_t is a count variable, with conditional mean $\mathrm{E}(y_t) = \exp(\boldsymbol{X}_t\boldsymbol{\beta})$ and conditional variance $\mathrm{E}\big(y_t - \exp(\boldsymbol{X}_t\boldsymbol{\beta})\big)^2 = \gamma^2\exp(\boldsymbol{X}_t\boldsymbol{\beta})$. Show that ML estimates of $\boldsymbol{\beta}$ under the incorrect assumption that y_t is generated by a Poisson regression model with mean $\exp(\boldsymbol{X}_t\boldsymbol{\beta})$ are asymptotically efficient in this case. Also show that the OLS covariance matrix from the artificial regression (11.55) is asymptotically valid.

Since the correct model has the form of a nonlinear regression model, we can obtain efficient estimates by the equivalent of weighted NLS. We want to make the vector of errors, $y_t - \exp(\boldsymbol{X}_t\boldsymbol{\beta})$, orthogonal to the derivatives of the regression function, which are given by the row vector $\exp(\boldsymbol{X}_t\boldsymbol{\beta})\boldsymbol{X}_t$, after both have been divided by the square root of the conditional variance. It follows that the moment conditions for asymptotically efficient estimation of this model are

$$\sum_{t=1}^{n} \frac{y_t - \exp(\boldsymbol{X}_t\boldsymbol{\beta})}{\gamma\exp(\frac{1}{2}\boldsymbol{X}_t\boldsymbol{\beta})}\,\frac{\exp(\boldsymbol{X}_t\boldsymbol{\beta})}{\gamma\exp(\frac{1}{2}\boldsymbol{X}_t\boldsymbol{\beta})}\boldsymbol{X}_t = \boldsymbol{0}.$$

Because the factors of $\exp(\frac{1}{2}\boldsymbol{X}_t\boldsymbol{\beta})$ in the two denominators cancel with the factor of $\exp(\boldsymbol{X}_t\boldsymbol{\beta})$ in the numerator of the second ratio, these conditions simplify to

$$\sum_{t=1}^{n} \frac{1}{\gamma^2}\big(y_t - \exp(\boldsymbol{X}_t\boldsymbol{\beta})\big)\boldsymbol{X}_t = \boldsymbol{0}. \tag{S11.41}$$

Since the factor of $1/\gamma^2$ has no effect on the solution, conditions (S11.41) are equivalent to the first-order conditions (11.51) which characterize the ML estimator for the Poisson regression model. Thus we see that the latter must be an asymptotically efficient estimator.

By standard results for weighted NLS estimation, we know that

$$\mathrm{Var}\Big(\operatorname*{plim}_{n\to\infty} n^{1/2}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)\Big) = \gamma^2\big(n^{-1}\boldsymbol{X}^{\top}\boldsymbol{\varUpsilon}(\boldsymbol{\beta})\boldsymbol{X}\big)^{-1}, \tag{S11.42}$$

where $\boldsymbol{\varUpsilon}(\boldsymbol{\beta})$ is an $n \times n$ diagonal matrix with typical diagonal element $\exp(\boldsymbol{X}_t\boldsymbol{\beta})$. If we run the artificial regression (11.55), with the regressand and regressors evaluated at the ML estimates $\hat{\boldsymbol{\beta}}$, the OLS covariance matrix is

$$s^2(\boldsymbol{X}^{\top}\hat{\boldsymbol{\varUpsilon}}\boldsymbol{X})^{-1}, \tag{S11.43}$$

where $\hat{\boldsymbol{\varUpsilon}} \equiv \boldsymbol{\varUpsilon}(\hat{\boldsymbol{\beta}})$, and

$$s^2 = \frac{1}{n-k}\sum_{t=1}^{n} \frac{\big(y_t - \exp(\boldsymbol{X}_t\hat{\boldsymbol{\beta}})\big)^2}{\exp(\boldsymbol{X}_t\hat{\boldsymbol{\beta}})}.$$

Since

$$\gamma^2 = \frac{\mathrm{E}\big(y_t - \exp(\boldsymbol{X}_t\boldsymbol{\beta})\big)^2}{\exp(\boldsymbol{X}_t\boldsymbol{\beta})},$$

and $\hat{\boldsymbol{\beta}}$ estimates $\boldsymbol{\beta}$ consistently, it is clear that s^2 estimates γ^2 consistently. Thus the OLS covariance matrix (S11.43) from the artificial regression has the same form as the asymptotic covariance matrix (S11.42), with the factor of n^{-1} inside the parentheses omitted and unknown parameters replaced by consistent estimates. It follows that (S11.43) is asymptotically valid.