Solution to Exercise 10.29

⋆10.29 Derive the loglikelihood function for the Box-Cox regression model (10.99). Then consider the following special case:

$$B(y_t, \lambda) = \beta_1 + \beta_2 B(x_t, \lambda) + u_t, \quad u_t \sim \mathrm{NID}(0, \sigma^2).$$

Derive the OPG regression for this model and explain precisely how to use it to test the hypotheses that the DGP is linear ($\lambda = 1$) and loglinear ($\lambda = 0$).

As usual, we start with the density of u_t, which is

$$(2\pi)^{-1/2}\sigma^{-1}\exp\bigl(-\tfrac{1}{2}u_t^2/\sigma^2\bigr).$$

We replace u_t by

$$B(y_t, \lambda) - \sum_{i=1}^{k_1}\beta_i z_{ti} - \sum_{i=k_1+1}^{k}\beta_i B(x_{ti}, \lambda)$$

and multiply by the Jacobian of the transformation, which is

$$\frac{\partial B(y_t, \lambda)}{\partial y_t} = y_t^{\lambda-1}. \tag{S10.79}$$

Therefore, the contribution to the loglikelihood made by observation t is

$$\begin{aligned}\ell_t(\boldsymbol{\beta}, \lambda, \sigma) = &-\tfrac{1}{2}\log 2\pi - \tfrac{1}{2}\log\sigma^2 + (\lambda - 1)\log y_t \\ &- \frac{1}{2\sigma^2}\Bigl(B(y_t, \lambda) - \sum_{i=1}^{k_1}\beta_i z_{ti} - \sum_{i=k_1+1}^{k}\beta_i B(x_{ti}, \lambda)\Bigr)^2.\end{aligned} \tag{S10.80}$$

The third term here is the Jacobian term, which is the the logarithm of (S10.79) and vanishes when $\lambda = 1$. The loglikelihood function is the sum of the contributions given by expression (S10.80) over all t:

$$\begin{aligned}\ell(\boldsymbol{\beta}, \lambda, \sigma) = &-\frac{n}{2}\log 2\pi - \frac{n}{2}\log\sigma^2 + (\lambda - 1)\sum_{t=1}^{n}\log y_t \\ &- \frac{1}{2\sigma^2}\sum_{t=1}^{n}\Bigl(B(y_t, \lambda) - \sum_{i=1}^{k_1}\beta_i x_{ti} - \sum_{i=k_1+1}^{k}\beta_i B(x_{ti}, \lambda)\Bigr)^2.\end{aligned} \tag{S10.81}$$

In the special case given in the question, the contribution to the loglikelihood made by the t^{th} observation simplifies to

$$C - \tfrac{1}{2}\log\sigma^2 + (\lambda - 1)\log y_t - \frac{1}{2\sigma^2}\bigl(B(y_t, \lambda) - \beta_1 - \beta_2 B(x_t, \lambda)\bigr)^2, \tag{S10.82}$$

where the constant $C \equiv -\frac{1}{2}\log 2\pi$ can be ignored for most purposes. The OPG regression has four regressors, each of which corresponds to one of the four parameters, β_1, β_2, λ, and σ. A typical element of each of these regressors is the derivative of (S10.82) with respect to the appropriate parameter. These derivatives are:

$$
\begin{aligned}
\beta_1\colon &\quad \frac{1}{\sigma^2}\big(B(y_t,\lambda)-\beta_1-\beta_2 B(x_t,\lambda)\big)\\
\beta_2\colon &\quad \frac{1}{\sigma^2}\big(B(y_t,\lambda)-\beta_1-\beta_2 B(x_t,\lambda)\big)B(x_t,\lambda)\\
\lambda\colon &\quad \log y_t-\frac{1}{\sigma^2}\big(B(y_t,\lambda)-\beta_1-\beta_2 B(x_t,\lambda)\big)\big(B'(y_t,\lambda)-\beta_2 B'(x_t,\lambda)\big)\\
\sigma\colon &\quad -\frac{1}{\sigma}+\frac{1}{\sigma^3}\big(B(y_t,\lambda)-\beta_1-\beta_2 B(x_t,\lambda)\big)^2
\end{aligned}
$$

In the expression for the regressor that corresponds to λ, $B'(z,\lambda)$ denotes the derivative of $B(z,\lambda)$ with respect to λ, which is

$$\frac{\lambda z^\lambda \log z - z^\lambda + 1}{\lambda^2}. \tag{S10.83}$$

For the OPG regression, the regressand is an n–vector of 1s, and the four regressors have the typical elements given above.

In order to test the hypothesis that the model is linear, that is, that $\lambda = 1$, we first regress y_t on a constant and x_t, obtaining parameter estimates under the null which we denote $\hat\beta_1$, $\hat\beta_2$, and $\hat\sigma$. We then evaluate the components of the OPG regression at these estimates and $\lambda = 1$. A typical observation of the OPG regression is

$$
\begin{aligned}
1 = {} & \frac{1}{\hat\sigma^2}\big(y_t-1-\hat\beta_1-\hat\beta_2(x_t-1)\big)c_1\\
& + \frac{1}{\hat\sigma^2}\big(y_t-1-\hat\beta_1-\hat\beta_2(x_t-1)\big)(x_t-1)c_2\\
& + \Big(\log y_t-\frac{1}{\hat\sigma^2}\big(y_t-1-\hat\beta_1-\hat\beta_2(x_t-1)\big)\big(B'(y_t,1)-\hat\beta_2 B'(x_t,1)\big)\Big)c_3\\
& + \Big(\frac{1}{\hat\sigma^3}\big(y_t-1-\hat\beta_1-\hat\beta_2(x_t-1)\big)^2-\frac{1}{\hat\sigma}\Big)c_4 + \text{residual},
\end{aligned}
$$

where, from (S10.83), we see that $B'(z,1) = z\log z - z + 1$. To test the null hypothesis that $\lambda = 1$, we can use $n - \mathrm{SSR}$ from this regression as a test statistic. It is asymptotically distributed as $\chi^2(1)$.

In order to test the hypothesis that the model is loglinear, that is, that $\lambda = 0$, we first regress $\log y_t$ on a constant and $\log x_t$, obtaining parameter estimates under the null which we denote $\tilde\beta_1$, $\tilde\beta_2$, and $\tilde\sigma$. We then evaluate the components of the OPG regression at these estimates and $\lambda = 0$. A typical observation of the OPG regression is

$$
\begin{aligned}
1 = {} & \frac{1}{\tilde\sigma^2}(\log y_t - \tilde\beta_1 - \tilde\beta_2 \log x_t)c_1 \\
& + \frac{1}{\tilde\sigma^2}(\log y_t - \tilde\beta_1 - \tilde\beta_2 \log x_t)\log x_t\, c_2 \\
& + \Big(\log y_t - \frac{1}{\tilde\sigma^2}(\log y_t - \tilde\beta_1 - \tilde\beta_2 \log x_t)\big(B'(y_t,0) - \tilde\beta_2 B'(x_t,0)\big)\Big)c_3 \\
& + \Big(\frac{1}{\tilde\sigma^3}(\log y_t - \tilde\beta_1 - \tilde\beta_2 \log x_t)^2 - \frac{1}{\tilde\sigma}\Big)c_4 + \text{residual}.
\end{aligned}
$$

In the above regression, we need an expression for $B'(z,1)$. If we attempt to evaluate expression (S10.83) at $\lambda = 0$, we find that both the numerator and the denominator equal 0. The first derivatives of both the numerator and the denominator are also equal to 0 when they are evaluated at $\lambda = 0$. However, the second derivatives at $\lambda = 0$ are equal to $(\log z)^2$ and 2, respectively. Thus l'Hôpital's Rule gives us the result that $B'(z,0) = \frac{1}{2}(\log z)^2$.

To test the hypothesis that $\lambda = 0$, we once again use $n - \text{SSR}$ from the OPG regression. This test statistic is asymptotically distributed as $\chi^2(1)$ under the null hypothesis.