Solution to Exercise 10.2

⋆10.2 Let the n–vector $\boldsymbol{y}$ be a vector of mutually independent realizations from the uniform distribution on the interval $[\beta_1, \beta_2]$, usually denoted by $\mathrm{U}(\beta_1, \beta_2)$. Thus, $y_t \sim \mathrm{U}(\beta_1, \beta_2)$ for $t = 1, \ldots, n$. Let $\hat{\beta}_1$ be the ML estimator of β_1 given in (10.13), and suppose that the true values of the parameters are $\beta_1 = 0$ and $\beta_2 = 1$. Show that the CDF of $\hat{\beta}_1$ is

$$F(\beta) \equiv \Pr(\hat{\beta}_1 \leq \beta) = 1 - (1 - \beta)^n.$$

Use this result to show that $n(\hat{\beta}_1 - \beta_{10})$, which in this case is just $n\hat{\beta}_1$, is asymptotically exponentially distributed with $\theta = 1$. Note that the PDF of the exponential distribution was given in (10.03). (**Hint:** The limit as $n \to \infty$ of $(1 + x/n)^n$, for arbitrary real x, is e^x.)

Show that, for arbitrary given β_{10} and β_{20}, with $\beta_{20} > \beta_{10}$, the asymptotic distribution of $n(\hat{\beta}_1 - \beta_{10})$ is characterized by the density (10.03) with $\theta = (\beta_{20} - \beta_{10})^{-1}$.

The probability that any single y_t is greater than β is $1 - \beta$. Thus, for n independent observations, the probability that every one of them is greater than β is $(1 - \beta)^n$. This is the probability that $\hat{\beta}_1$, the smallest y_t, must be greater than β. Therefore,

$$\Pr(\hat{\beta}_1 \leq \beta) = 1 - (1 - \beta)^n, \tag{S10.04}$$

as we were asked to show.

For the second part, let Z denote $n\hat{\beta}_1$ and let z denote $n\beta$. Since

$$\Pr(Z \leq z) = \Pr(\hat{\beta}_1 \leq \beta),$$

we have just shown that the CDF of Z is

$$F(z) = 1 - (1 - z/n)^n. \tag{S10.05}$$

By the result alluded to in the hint, the limit of this as $n \to \infty$ is

$$F^{\infty}(z) \equiv \lim_{n\to\infty} F(z) = 1 - e^{-z}, \tag{S10.06}$$

which is the CDF of the exponential distribution with $\theta = 1$. The PDF that corresponds to this CDF is then

$$f^{\infty}(z) = -\frac{\partial e^{-z}}{\partial z} = e^{-z},$$

which is just a special case of (10.03) with $\theta = 1$.

When β_{10} and β_{20} are arbitrary, the probability that any single y_t is greater than β is $1 - \theta(\beta - \beta_{10})$, where $\theta \equiv (\beta_{20} - \beta_{10})^{-1}$. Therefore, the equivalent of equation (S10.04) is

$$\Pr(\hat{\beta}_1 \le \beta) = 1 - \big(1 - \theta(\beta - \beta_{10})\big)^n,$$

and the equivalent of equation (S10.05) is

$$F(z) = 1 - (1 - \theta z/n)^n.$$

Here we have redefined Z as $n(\hat{\beta}_1 - \beta_{10})$. By the same result that led to (S10.06), we find that

$$F^{\infty}(z) \equiv \lim_{n\to\infty} F(z) = 1 - e^{-\theta z},$$

which is the CDF of the exponential distribution with arbitrary θ. The PDF that corresponds to this CDF is then

$$f^{\infty}(z) = -\frac{\partial e^{-\theta z}}{\partial z} = \theta e^{-\theta z},$$

which is the density (10.03), namely, the PDF of the exponential distribution. Since we have already defined θ as $(\beta_{20} - \beta_{10})^{-1}$, this is the result we were required to show.