Solution to Exercise 10.19

⋆10.19 Let the loglikelihood function $\ell(\theta)$ depend on one scalar parameter θ. For this special case, consider the distribution of the LM statistic (10.69) under the drifting DGP characterized by the parameter $\theta = n^{-1/2}\delta$ for a fixed δ. This DGP drifts toward the fixed DGP with $\theta = 0$, which we think of as representing the null hypothesis. Show first that $n^{-1}\boldsymbol{I}(n^{-1/2}\delta) \to \mathcal{I}(0)$ as $n \to \infty$. Here the asymptotic information matrix $\mathcal{I}(\theta)$ is just a scalar, since there is only one parameter.

Next, show that $n^{-1/2}$ times the gradient, evaluated at $\theta = 0$, which we may write as $n^{-1/2}g(0)$, is asymptotically normally distributed with mean $\delta\,\mathcal{I}(0)$ and variance $\mathcal{I}(0)$. Finally, show that the LM statistic is asymptotically distributed as $\chi^2(1)$ with a finite noncentrality parameter, and give the value of that noncentrality parameter.

We see by a Taylor expansion of the definition (10.31) that

$$
\begin{aligned}
n^{-1}\boldsymbol{I}(n^{-1/2}\delta) &= \frac{1}{n}\sum_{t=1}^{n}\boldsymbol{I}_t(n^{-1/2}\delta) \\
&= \frac{1}{n}\sum_{t=1}^{n}\boldsymbol{I}_t(0) + \frac{1}{n}\sum_{t=1}^{n}\boldsymbol{I}_t'(n^{-1/2}\bar{\delta})n^{-1/2}\delta,
\end{aligned}
$$

where $\boldsymbol{I}_t'(\theta)$ is the derivative of the contribution $\boldsymbol{I}_t(\theta)$, which is just a scalar function of a scalar argument, and $0 \le \bar{\delta} \le \delta$. Taking the limit as $n \to \infty$ of the above relation gives the first result, since $\lim n^{-1}\sum \boldsymbol{I}_t(0) = \mathcal{I}(0)$ by definition, and $n^{-1}\sum \boldsymbol{I}_t'(n^{-1/2}\bar{\delta})$ is bounded above as $n \to \infty$.

Another Taylor expansion, of $n^{-1/2}g(n^{-1/2}\delta)$ this time, gives

$$
n^{-1/2}g(n^{-1/2}\delta) = n^{-1/2}g(0) + n^{-1}H(n^{-1/2}\bar{\delta})\delta, \tag{S10.44}
$$

where $H(\theta)$ is the Hessian, which is also just a scalar in this special case. For each value of n, the true value of θ is $n^{-1/2}\delta$, and so the expectation of $n^{-1/2}g(n^{-1/2}\delta)$ is zero for all n. Similarly, the variance of $n^{-1/2}g(n^{-1/2}\delta)$ is $n^{-1}\boldsymbol{I}(n^{-1/2}\delta)$. Thus the plim of $n^{-1/2}g(n^{-1/2}\delta)$ has expectation 0 and variance the limit of $n^{-1}\boldsymbol{I}(n^{-1/2}\delta)$, which is $\mathcal{I}(0)$ by the first part of the exercise. A central limit theorem can be used to show that the plim is also asymptotically normal. Thus we see that the limit of the left-hand side of (S10.44) is a variable distributed as $\mathrm{N}\big(0, \mathcal{I}(0)\big)$

The argument used in first part of the exercise shows that the limit of $n^{-1}H(n^{-1/2}\delta)$ as $n \to \infty$ is $\mathcal{H}(0)$, which is equal to $-\mathcal{I}(0)$ by the information matrix equality. Thus the limit of the second term on the right-hand side of (S10.44) is the deterministic quantity $-\delta\mathcal{I}(0)$. It follows that the plim of $n^{-1/2}g(0)$, the first term on the right-hand side of (S10.44), is distributed as $\mathrm{N}\big(\delta\mathfrak{I}(0), \mathfrak{I}(0)\big)$, as we were asked to show.

The LM statistic (10.69) for the null hypothesis $\theta = 0$ can be written as

$$\mathrm{LM} = \frac{g^2(0)}{\boldsymbol{I}(0)} = \left(\frac{n^{-1/2}g(0)}{\big(n^{-1}\boldsymbol{I}(0)\big)^{1/2}}\right)^2.$$

The random variable in the numerator of the rightmost expression here has a plim that is distributed as $\mathrm{N}\big(\delta\mathfrak{I}(0), \mathfrak{I}(0)\big)$. When this variable is divided by the square root of its variance, the result is asymptotically distributed as $\mathrm{N}\big(\delta\mathfrak{I}^{1/2}(0), 1\big)$. The LM statistic, which is the square of this result, is therefore asymptotically distributed as noncentral $\chi^2(1)$ with noncentrality parameter $\delta^2\mathfrak{I}(0)$; see Section 4.7.