Solution to Exercise 1.14

⋆1.14 Let a random variable X_1 be distributed as $N(0, 1)$. Now suppose that a second random variable, X_2, is constructed as the product of X_1 and an independent random variable Z, which equals 1 with probability $1/2$ and -1 with probability $1/2$.

What is the (marginal) distribution of X_2? What is the covariance between X_1 and X_2? What is the distribution of X_1 conditional on X_2?

The marginal distribution of X_2 is just the standard normal distribution. Since the standard normal density is symmetric, randomly replacing X_1 with $-X_1$ half (or any fraction) of the time does not change the distribution at all. Formally,

$$\begin{aligned}\Pr(X_2 \leq x) &= \Pr\big((Z=1)\cap(X_1 \leq x)\big) + \Pr\big((Z=-1)\cap(X_1 \geq -x)\big)\\ &= \tfrac{1}{2}\big(\Phi(x) + 1 - \Phi(-x)\big)\\ &= \tfrac{1}{2}\big(\Phi(x) + \Phi(x)\big) = \Phi(x).\end{aligned}$$

The step to the second line above follows from the independence of X_1 and Z, and the fact that $\Pr(X_1 \geq x) = 1 - \Pr(X_1 \leq x)$. The step to the third line uses the symmetry of the standard normal distribution, which implies that $\Pr(X_1 \leq x) = \Pr(X_1 \geq -x)$, that is, $\Phi(x) = 1 - \Phi(-x)$.

The covariance between X_1 and X_2 is clearly zero. Since both X_1 and X_2 have mean zero, we do not need to subtract the expectations when we compute the covariance. Thus we have

$$\mathrm{Cov}(X_1, X_2) = \mathrm{E}(X_1 X_2) = \tfrac{1}{2}\mathrm{E}(X_1^2) - \tfrac{1}{2}\mathrm{E}(X_1^2) = 0.$$

The distribution of X_1 conditional on X_2 is exactly the same as the distribution of X_2 conditional on X_1. That is, $X_1 = X_2$ with probability $1/2$, and $X_1 = -X_2$ with probability $1/2$.