Solution to Exercise 1.14

**1.14** Let a random variable $X_1$ be distributed as $N(0,1)$. Now suppose that a second random variable, $X_2$, is constructed as the product of $X_1$ and an independent random variable $Z$, which equals 1 with probability $1/2$ and $-1$ with probability $1/2$.

What is the (marginal) distribution of $X_2$? What is the covariance between $X_1$ and $X_2$? What is the distribution of $X_1$ conditional on $X_2$?

The marginal distribution of $X_2$ is just the standard normal distribution. Since the standard normal density is symmetric, randomly replacing $X_1$ with $-X_1$ half (or any fraction) of the time does not change the distribution at all. Formally,

$$
\Pr(X_2 \leq x) = \Pr((Z = 1) \cap (X_1 \leq x)) + \Pr((Z = -1) \cap (X_1 \geq -x))
$$

$$
= \frac{1}{2} (\Phi(x) + 1 - \Phi(-x))
$$

$$
= \frac{1}{2} (\Phi(x) + \Phi(x)) = \Phi(x).
$$

The step to the second line above follows from the independence of $X_1$ and $Z$, and the fact that $\Pr(X_1 \geq x) = 1 - \Pr(X_1 \leq x)$. The step to the third line uses the symmetry of the standard normal distribution, which implies that $\Pr(X_1 \leq x) = \Pr(X_1 \geq -x)$, that is, $\Phi(x) = 1 - \Phi(-x)$.

The covariance between $X_1$ and $X_2$ is clearly zero. Since both $X_1$ and $X_2$ have mean zero, we do not need to subtract the expectations when we compute the covariance. Thus we have

$$
\text{Cov}(X_1, X_2) = \text{E}(X_1X_2) = \frac{1}{2} \text{E}(X_1^2) - \frac{1}{2} \text{E}(X_1^2) = 0.
$$

The distribution of $X_1$ conditional on $X_2$ is exactly the same as the distribution of $X_2$ conditional on $X_1$. That is, $X_1 = X_2$ with probability $1/2$, and $X_1 = -X_2$ with probability $1/2$. 

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