Solution to Exercise 1.12

$\star$**1.12** Show that the variance of the random variable $X_1 - \mathrm{E}(X_1 \,|\, X_2)$ cannot be greater than the variance of X_1, and that the two variances are equal if X_1 and X_2 are independent. This result shows how one random variable can be informative about another: Conditioning on it reduces variance unless the two variables are independent.

We assume to begin with that $\mathrm{E}(X_1) = 0$. Then the variance of X_1 is just $\mathrm{E}(X_1^2)$. This can be written as

$$\begin{aligned}
&\mathrm{E}\Big(\big(X_1 - \mathrm{E}(X_1 \,|\, X_2) + \mathrm{E}(X_1 \,|\, X_2)\big)^2\Big) \\
&\quad = \mathrm{E}\Big(\big(X_1 - \mathrm{E}(X_1 \,|\, X_2)\big)^2\Big) + 2\mathrm{E}\Big(\big(X_1 - \mathrm{E}(X_1 \,|\, X_2)\big)\,\mathrm{E}(X_1 \,|\, X_2)\Big) \\
&\qquad + \mathrm{E}\big(\mathrm{E}(X_1 \,|\, X_2)^2\big).
\end{aligned}$$

The first term on the right-hand side above is the variance of $X_1 - \mathrm{E}(X_1 \,|\, X_2)$. The middle term is the covariance of $X_1 - \mathrm{E}(X_1 \,|\, X_2)$ and $\mathrm{E}(X_1 \,|\, X_2)$, which is zero, as we showed in the solution to the previous exercise. The third term is the variance of $\mathrm{E}(X_1 \,|\, X_2)$. Thus we have

$$\mathrm{Var}(X_1) = \mathrm{Var}\big(X_1 - \mathrm{E}(X_1 \,|\, X_2)\big) + \mathrm{Var}\big(\mathrm{E}(X_1 \,|\, X_2)\big). \tag{S1.09}$$

Since a variance is necessarily nonnegative, it follows that

$$\mathrm{Var}(X_1) \geq \mathrm{Var}\big(X_1 - \mathrm{E}(X_1 \,|\, X_2)\big). \tag{S1.10}$$

If X_1 and X_2 are independent, then $\mathrm{E}(X_1 \,|\, X_2) = \mathrm{E}(X_1) = 0$. In this case, it follows that $X_1 = X_1 - \mathrm{E}(X_1 \,|\, X_2)$, and so the variance of X_1 is the same as that of $X_1 - \mathrm{E}(X_1 \,|\, X_2)$.

If $\mathrm{E}(X_1) = \mu_1 \neq 0$, we can apply (S1.10) to $X_1 - \mu_1$, which has mean zero. Since a variance is a *central* moment, we have that $\mathrm{Var}(X_1) = \mathrm{Var}(X_1 - \mu_1)$. The right-hand side of (S1.10) for $X_1 - \mu_1$ is

$$\begin{aligned}
\mathrm{Var}\big(X_1 - \mu_1 - \mathrm{E}(X_1 - \mu_1 \,|\, X_2)\big) &= \mathrm{Var}\big(X_1 - \mu_1 - \mathrm{E}(X_1 \,|\, X_2) + \mu_1\big) \\
&= \mathrm{Var}\big(X_1 - \mathrm{E}(X_1 \,|\, X_2)\big),
\end{aligned}$$

from which we may conclude that both sides of (S1.10) are independent of the value of μ_1, and so (S1.10) is true whether X_1 has mean zero or not.