Solution to Exercise 1.11

⋆1.11 Show that the covariance of the random variable $\mathrm{E}(X_1 \,|\, X_2)$ and the random variable $X_1 - \mathrm{E}(X_1 \,|\, X_2)$ is zero. It is easiest to show this result by first showing that it is true when the covariance is computed conditional on X_2.

First, observe that

$$\mathrm{E}\big(X_1 - \mathrm{E}(X_1 \,|\, X_2) \,|\, X_2\big) = \mathrm{E}(X_1 \,|\, X_2) - \mathrm{E}(X_1 \,|\, X_2) = 0. \tag{S1.07}$$

Thus the covariance of $\mathrm{E}(X_1 \,|\, X_2)$ and $X_1 - \mathrm{E}(X_1 \,|\, X_2)$, conditional on X_2, is the expectation, conditional on X_2, of the product

$$\mathrm{E}(X_1 \,|\, X_2)\,\big(X_1 - \mathrm{E}(X_1 \,|\, X_2)\big). \tag{S1.08}$$

The first factor, being an expectation conditional on X_2, is a deterministic function of X_2; see equation (1.17). Thus the expectation of (S1.08) conditional on X_2 is

$$\mathrm{E}(X_1 \,|\, X_2)\,\mathrm{E}\big(X_1 - \mathrm{E}(X_1 \,|\, X_2) \,|\, X_2\big) = 0.$$

The equality holds because, by (S1.07), the second factor is zero. The Law of Iterated Expectations now tells us that the unconditional expectation of expression (S1.08), which is the unconditional covariance of $\mathrm{E}(X_1 \,|\, X_2)$ and $X_1 - \mathrm{E}(X_1 \,|\, X_2)$, is zero as well.