only one restriction is to be tested, a suitable test statistic is the t statistic for the artificial parameter b_ρ in (6.84) to equal 0. This is the square root of the F statistic, which we have seen to be asymptotically valid.

Almost as simple as the above test is a test of the null hypothesis (6.82) against an alternative in which the error terms follow the **AR(2) process**

$$u_t = \rho_1 u_{t-1} + \rho_2 u_{t-2} + \varepsilon_t, \quad \varepsilon_t \sim \mathrm{IID}(0, \sigma^2).$$

It is not hard to show that an appropriate artificial regression for testing (6.82) against the AR(2) alternative that is analogous to (6.06) is

$$\tilde{u}_t = \boldsymbol{X}_t \boldsymbol{b} + b_{\rho_1} \tilde{u}_{t-1} + b_{\rho_2} \tilde{u}_{t-2} + \text{residual}; \tag{6.85}$$

see Exercise 6.10. Since, in this case, we have a test with two degrees of freedom, we cannot use a t test. However, it is still not necessary to run two regressions in order to compute an F statistic. Consider the form taken by GNR_0 in this case:

$$\tilde{u}_t = \boldsymbol{X}_t \boldsymbol{b} + \text{residual}. \tag{6.86}$$

This is just the GNR corresponding to the linear regression (6.82). Since the regressand is the vector of residuals from estimating (6.82), it is orthogonal to the explanatory variables. Therefore, by (6.55), the artificial parameter estimates $\tilde{\boldsymbol{b}}$ are zero, and (6.86) has no explanatory power. As a result, the SSR from (6.86) is equal to the total sum of squares (TSS). But this is also the TSS from the GNR (6.85) corresponding to the alternative. Thus the difference between the SSRs from (6.86) and (6.85) is the difference between the TSS and the SSR from (6.85), or, more conveniently, the explained sum of squares (ESS) from (6.85). The GNR-based F statistic can therefore be computed by running (6.85) alone. In fact, since the denominator is just the estimate $\tilde{s}^2$ of the error variance from (6.85), the F statistic is simply[1]

$$F = \frac{\mathrm{ESS}}{r\tilde{s}^2} = \frac{n-k-r}{r} \times \frac{\mathrm{ESS}}{\mathrm{SSR}}, \tag{6.87}$$

where k is the number of regressors in (6.82) and $r = 2$ in this particular case.

Asymptotically, we can obtain a valid test statistic by using any consistent estimate of the true error variance σ_0^2 as the denominator. If we were to use the estimate under the null rather than the estimate under the alternative, the denominator of the test statistic would be $(n-k_1)^{-1}\sum_{t=1}^n \tilde{u}_t^2$. Asymptotically, it makes no difference whether we divide by $n - k_1$ or n when we estimate σ^2.

[1] We are assuming here that regression (6.85) is run over all n observations. This requires either that data for observations 0 and -1 are available, or that the unobserved residuals $\tilde{u}_0$ and $\tilde{u}_{-1}$ are replaced by zeros.