are generated by (4.47) with $\beta_2 = 0$, we have that $\boldsymbol{M}_1\boldsymbol{y} = \boldsymbol{M}_1\boldsymbol{u}$, and so (4.50) is asymptotically equivalent to

$$\frac{n^{-1/2}\boldsymbol{x}_2^\top\boldsymbol{M}_1\boldsymbol{u}}{\sigma_0(n^{-1}\boldsymbol{x}_2^\top\boldsymbol{M}_1\boldsymbol{x}_2)^{1/2}}. \tag{4.51}$$

It is now easy to derive the asymptotic distribution of t_{β_2} if for a moment we reinstate the assumption that the regressors are exogenous. In that case, we can work conditionally on $\boldsymbol{X}$, which means that the only part of (4.51) that is treated as random is $\boldsymbol{u}$. The numerator of (4.51) is $n^{-1/2}$ times a weighted sum of the u_t, each of which has mean 0, and the conditional variance of this weighted sum is

$$\mathrm{E}(\boldsymbol{x}_2^\top\boldsymbol{M}_1\boldsymbol{u}\boldsymbol{u}^\top\boldsymbol{M}_1\boldsymbol{x}_2 \,|\, \boldsymbol{X}) = \sigma_0^2\boldsymbol{x}_2^\top\boldsymbol{M}_1\boldsymbol{x}_2.$$

Thus (4.51) evidently has mean 0 and variance 1, conditional on $\boldsymbol{X}$. But since 0 and 1 do not depend on $\boldsymbol{X}$, these are also the unconditional mean and variance of (4.51). Provided that we can apply a CLT to the numerator of (4.51), the numerator of t_{β_2} must be asymptotically normally distributed, and we conclude that, under the null hypothesis, with exogenous regressors,

$$t_{\beta_2} \overset{a}{\sim} \mathrm{N}(0,1). \tag{4.52}$$

The notation "$\overset{a}{\sim}$" means that t_{β_2} is **asymptotically distributed** as $\mathrm{N}(0,1)$. Since the DGP is assumed to be (4.47), this result does *not* require that the error terms be normally distributed.

The t Test with Predetermined Regressors

If we relax the assumption of exogenous regressors, the analysis becomes more complicated. Readers not interested in the algebraic details may well wish to skip to the next section, since what follows is not essential for understanding the rest of this chapter. However, this subsection provides an excellent example of how asymptotic theory works, and it illustrates clearly just why we can relax some assumptions but not others.

We begin by applying a CLT to the k-vector

$$\boldsymbol{v} \equiv n^{-1/2}\boldsymbol{X}^\top\boldsymbol{u} = n^{-1/2}\sum_{t=1}^{n} u_t\boldsymbol{X}_t^\top. \tag{4.53}$$

By assumption (3.10), $\mathrm{E}(u_t \,|\, \boldsymbol{X}_t) = 0$. This implies that $\mathrm{E}(u_t\boldsymbol{X}_t^\top) = \boldsymbol{0}$, as required for the CLT, which then tells us that

$$\boldsymbol{v} \overset{a}{\sim} \mathrm{N}\Big(\boldsymbol{0}, \lim_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}\mathrm{Var}(u_t\boldsymbol{X}_t^\top)\Big) = \mathrm{N}\Big(\boldsymbol{0}, \lim_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}\mathrm{E}(u_t^2\boldsymbol{X}_t^\top\boldsymbol{X}_t)\Big);$$