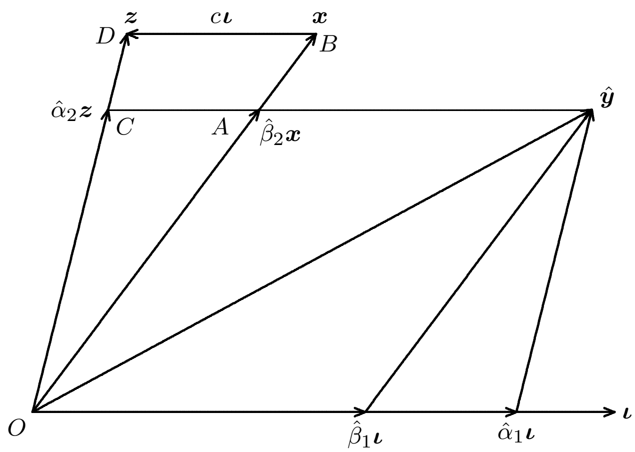

Figure 2.12 Adding a constant does not affect the slope coefficient

The effect on y_t of a change of one unit in x_t is measured by the slope coefficient β_2. If we hold $\bar{x}$ at its value before x_t is changed, then the unit change in x_t induces a unit change in z_t. Thus a unit change in z_t, which is measured by the slope coefficient α_2, should have the same effect as a unit change in x_t. Accordingly, $\alpha_2 = \beta_2$, just as we found above.

The slope coefficients α_2 and β_2 would be the same with any constant in the place of $\bar{x}$. The reason for this can be seen geometrically, as illustrated in Figure 2.12. This figure, which is constructed in the same way as panel (b) of Figure 2.11, depicts the span of $\boldsymbol{\iota}$ and $\boldsymbol{x}$, with $\boldsymbol{\iota}$ in the horizontal direction. As before, the vector $\boldsymbol{y}$ is not shown, because a third dimension would be required; the vector would extend from the origin to a point off the plane of the page and directly above (or below) the point labelled $\hat{\boldsymbol{y}}$.

The figure shows the vector of fitted values $\hat{\boldsymbol{y}}$ as the vector sum $\hat{\beta}_1\boldsymbol{\iota} + \hat{\beta}_2\boldsymbol{x}$. The slope coefficient $\hat{\beta}_2$ is the ratio of the length of the vector $\hat{\beta}_2\boldsymbol{x}$ to that of $\boldsymbol{x}$; geometrically, it is given by the ratio OA/OB. Then a new regressor $\boldsymbol{z}$ is defined by adding the constant value c, which is negative in the figure, to each component of $\boldsymbol{x}$, giving $\boldsymbol{z} = \boldsymbol{x} + c\boldsymbol{\iota}$. In terms of this new regressor, the vector $\hat{\boldsymbol{y}}$ is given by $\hat{\alpha}_1\boldsymbol{\iota} + \hat{\alpha}_2\boldsymbol{z}$, and $\hat{\alpha}_2$ is given by the ratio OC/OD. Since the ratios OA/OB and OC/OD are clearly the same, we see that $\hat{\alpha}_2 = \hat{\beta}_2$. A formal argument would use the fact that OAC and OBD are similar triangles.

When the constant c is chosen as $\bar{x}$, the vector $\boldsymbol{z}$ is said to be centered, and, as we saw above, it is orthogonal to $\boldsymbol{\iota}$. In this case, the estimate $\hat{\alpha}_2$ is the same whether it is obtained by regressing $\boldsymbol{y}$ on both $\boldsymbol{\iota}$ and $\boldsymbol{z}$, or just on $\boldsymbol{z}$ alone. This is illustrated in Figure 2.13, which shows what Figure 2.12 would look like when $\boldsymbol{z}$ is orthogonal to $\boldsymbol{\iota}$. Once again, the vector of fitted values $\hat{\boldsymbol{y}}$ is decomposed as $\hat{\alpha}_1\boldsymbol{\iota} + \hat{\alpha}_2\boldsymbol{z}$, with $\boldsymbol{z}$ now at right angles to $\boldsymbol{\iota}$.

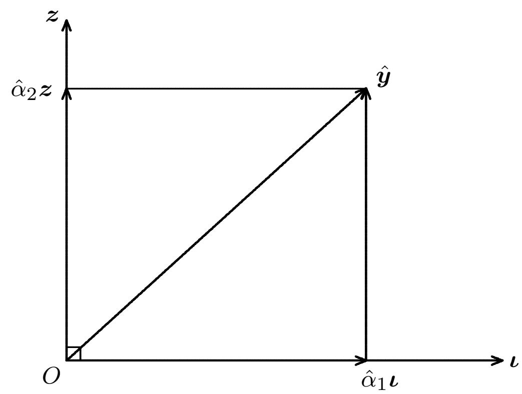

Figure 2.13 Orthogonal regressors may be omitted

Now suppose that $\boldsymbol{y}$ is regressed on $\boldsymbol{z}$ alone. This means that $\boldsymbol{y}$ is projected orthogonally on to $\mathcal{S}(\boldsymbol{z})$, which in the figure is the vertical line through $\boldsymbol{z}$. By definition,

$$\boldsymbol{y} = \hat{\alpha}_1 \boldsymbol{\iota} + \hat{\alpha}_2 \boldsymbol{z} + \hat{\boldsymbol{u}}, \tag{2.32}$$

where $\hat{\boldsymbol{u}}$ is orthogonal to both $\boldsymbol{\iota}$ and $\boldsymbol{z}$. But $\boldsymbol{\iota}$ is also orthogonal to $\boldsymbol{z}$, and so the only term on the right-hand side of (2.32) not to be annihilated by the projection on to $\mathcal{S}(\boldsymbol{z})$ is the middle term, which is left unchanged by it. Thus the fitted value vector from regressing $\boldsymbol{y}$ on $\boldsymbol{z}$ alone is just $\hat{\alpha}_2 \boldsymbol{z}$, and so the OLS estimate is the same $\hat{\alpha}_2$ as given by the regression on both $\boldsymbol{\iota}$ and $\boldsymbol{z}$. Geometrically, we obtain this result because the projection of $\boldsymbol{y}$ on to $\mathcal{S}(\boldsymbol{z})$ is the same as the projection of $\hat{\boldsymbol{y}}$ on to $\mathcal{S}(\boldsymbol{z})$.

Incidentally, the fact that OLS residuals are orthogonal to all the regressors, including $\boldsymbol{\iota}$, leads to the important result that the residuals in any regression with a constant term sum to zero. In fact,

$$\boldsymbol{\iota}^\top \hat{\boldsymbol{u}} = \sum_{t=1}^{n} \hat{u}_t = 0;$$

recall equation (1.29). The residuals also sum to zero in any regression for which $\boldsymbol{\iota} \in \mathcal{S}(\boldsymbol{X})$, even if $\boldsymbol{\iota}$ does not explicitly appear in the list of regressors. This can happen if the regressors include certain sets of **dummy variables**, as we will see in Section 2.5.

Two Groups of Regressors

The results proved in the previous subsection are actually special cases of more general results that apply to any regression in which the regressors can logically be broken up into two groups. Such a regression can be written as

$$\boldsymbol{y} = \boldsymbol{X}_1 \boldsymbol{\beta}_1 + \boldsymbol{X}_2 \boldsymbol{\beta}_2 + \boldsymbol{u}, \tag{2.33}$$