$\sigma_t^2 = y_t$, and $u_t^2 = \varepsilon_t$, we see that the former becomes formally the same as an ARMA(p, q) process in which the coefficient of ε_t equals 0. However, the formal similarity between the two processes masks some important differences. In a GARCH process, the σ_t^2 are not observable, and $\mathrm{E}(u_t^2 \,|\, \Omega_t) = \sigma_t^2 \neq 0$.

The simplest and by far the most popular GARCH model is the **GARCH(1,1) process**, for which the conditional variance can be written as

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \delta_1 \sigma_{t-1}^2. \tag{13.78}$$

Under the hypothesis of covariance stationarity, the unconditional variance σ^2 can be found by taking the unconditional expectation of equation (13.78). We find that

$$\sigma^2 = \alpha_0 + \alpha_1 \sigma^2 + \delta_1 \sigma^2.$$

Solving this equation yields the result that

$$\sigma^2 = \frac{\alpha_0}{1 - \alpha_1 - \delta_1}. \tag{13.79}$$

For this unconditional variance to exist, it must be the case that $\alpha_1 + \delta_1 < 1$, and for it to be positive, we require that $\alpha_0 > 0$.

The GARCH$(1, 1)$ process generally seems to work quite well in practice. In many cases, it cannot be rejected against any more general GARCH(p, q) process. An interesting empirical regularity is that the estimate $\hat{\alpha}_1$ is often small and positive, with the estimate $\hat{\delta}_1$ much larger, and the sum of the coefficients, $\hat{\alpha}_1 + \hat{\delta}_1$, between 0.9 and 1. These parameter values imply that the time-varying volatility is highly persistent.

Testing for ARCH Errors

It is easy to test a regression model for the presence of ARCH or GARCH errors. Imagine, for the moment, that we actually observe the u_t. Then we can replace σ_t^2 by $u_t^2 - e_t$, where e_t is defined to be the difference between u_t^2 and its conditional expectation. This allows us to rewrite the GARCH(p, q) model (13.76) as

$$u_t^2 = \alpha_0 + \sum_{i=1}^{\max(p,q)} (\alpha_i + \delta_i) u_{t-i}^2 + e_t - \sum_{j=1}^{p} \delta_j \, e_{t-j}. \tag{13.80}$$

In this equation, we have replaced all of the σ_{t-j}^2 by $u_{t-j}^2 - e_{t-j}$ and then grouped the two summations that involve the u_{t-i}^2. Of course, if $p \neq q$, either some of the α_i or some of the δ_i in the first summation are identically zero. Equation (13.80) can now be interpreted as a regression model with dependent variable u_t^2 and MA(p) errors. If one were actually to estimate (13.80), the MA structure would yield estimates of the δ_j, and the estimated coefficients of the u_{t-i}^2 would then allow the α_i to be estimated.