

Similarly, it can be shown that the Hessian $\mathbf{H}(\boldsymbol{\beta})$ has typical element

$$H_{ij}(\boldsymbol{\beta}) = -\frac{2}{n} \sum_{t=1}^n \left((y_t - x_t(\boldsymbol{\beta})) \frac{\partial X_{ti}(\boldsymbol{\beta})}{\partial \beta_j} - X_{ti}(\boldsymbol{\beta}) X_{tj}(\boldsymbol{\beta}) \right). \quad (6.46)$$

When this expression is evaluated at $\boldsymbol{\beta}_0$, it is asymptotically equivalent to

$$\frac{2}{n} \sum_{t=1}^n X_{ti}(\boldsymbol{\beta}_0) X_{tj}(\boldsymbol{\beta}_0). \quad (6.47)$$

The reason for this asymptotic equivalence is that, since $y_t = x_t(\boldsymbol{\beta}_0) + u_t$, the first term inside the large parentheses in (6.46) becomes

$$-\frac{2}{n} \sum_{t=1}^n \frac{\partial X_{ti}(\boldsymbol{\beta})}{\partial \beta_j} u_t. \quad (6.48)$$

Because $x_t(\boldsymbol{\beta})$ and all its first- and second-order derivatives belong to Ω_t , the expectation of each term in (6.48) is 0. Therefore, by a law of large numbers, expression (6.48) tends to 0 as $n \rightarrow \infty$.

Gauss-Newton Methods

The above results make it clear that a natural choice for $\mathbf{D}(\boldsymbol{\beta})$ in a quasi-Newton minimization algorithm based on (6.43) is

$$\mathbf{D}(\boldsymbol{\beta}) = 2n^{-1} \mathbf{X}^\top(\boldsymbol{\beta}) \mathbf{X}(\boldsymbol{\beta}). \quad (6.49)$$

By construction, this $\mathbf{D}(\boldsymbol{\beta})$ is positive definite whenever $\mathbf{X}(\boldsymbol{\beta})$ has full rank. Substituting equations (6.49) and (6.45) into equation (6.43) yields

$$\begin{aligned} \boldsymbol{\beta}_{(j+1)} &= \boldsymbol{\beta}_{(j)} + \alpha_{(j)} (2n^{-1} \mathbf{X}_{(j)}^\top \mathbf{X}_{(j)})^{-1} (2n^{-1} \mathbf{X}_{(j)}^\top (\mathbf{y} - \mathbf{x}_{(j)})) \\ &= \boldsymbol{\beta}_{(j)} + \alpha_{(j)} (\mathbf{X}_{(j)}^\top \mathbf{X}_{(j)})^{-1} \mathbf{X}_{(j)}^\top (\mathbf{y} - \mathbf{x}_{(j)}). \end{aligned} \quad (6.50)$$

The classic **Gauss-Newton method** would set $\alpha_{(j)} = 1$, so that

$$\boldsymbol{\beta}_{(j+1)} = \boldsymbol{\beta}_{(j)} + (\mathbf{X}_{(j)}^\top \mathbf{X}_{(j)})^{-1} \mathbf{X}_{(j)}^\top (\mathbf{y} - \mathbf{x}_{(j)}), \quad (6.51)$$

but it is generally better to use a good one-dimensional search routine to choose α optimally at each iteration. This modified type of Gauss-Newton procedure often works quite well in practice.

The second term on the right-hand side of (6.51) can most easily be computed by means of an **artificial regression** called the **Gauss-Newton regression**, or **GNR**. This artificial regression can be expressed as follows:

$$\mathbf{y} - \mathbf{x}(\boldsymbol{\beta}) = \mathbf{X}(\boldsymbol{\beta}) \mathbf{b} + \text{residuals}. \quad (6.52)$$