**4.10** Show that the OLS estimates $\tilde{\boldsymbol{\beta}}_1$ from the restricted model (4.29) can be obtained from those of the unrestricted model (4.28) by the formula

$$\tilde{\boldsymbol{\beta}}_1 = \hat{\boldsymbol{\beta}}_1 + (\boldsymbol{X}_1^{\top}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1^{\top}\boldsymbol{X}_2\hat{\boldsymbol{\beta}}_2.$$

**Hint:** Equation (4.38) is useful for this exercise.

**4.11** Consider regressions (4.42) and (4.41), which are numerically equivalent. You may drop the normality assumption and assume that the error terms are merely IID. Show that the SSR from these regressions is equal to the sum of the SSRs from the two subsample regressions:

$$\begin{aligned} \boldsymbol{y}_1 &= \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{u}_1, \quad \boldsymbol{u}_1 \sim \mathrm{IID}(\mathbf{0}, \sigma^2\mathbf{I}), \text{ and} \\ \boldsymbol{y}_2 &= \boldsymbol{X}_2\boldsymbol{\beta}_2 + \boldsymbol{u}_2, \quad \boldsymbol{u}_2 \sim \mathrm{IID}(\mathbf{0}, \sigma^2\mathbf{I}). \end{aligned}$$

**4.12** When performing a Chow test, one may find that one of the subsamples is smaller than $k$, the number of regressors. Without loss of generality, assume that $n_2 < k$. Show that, in this case, the $F$ statistic becomes

$$\frac{(\mathrm{RSSR} - \mathrm{SSR}_1)/n_2}{\mathrm{SSR}_1/(n_1 - k)},$$

and that the numerator and denominator really have the degrees of freedom used in this formula.

**4.13** Show, using the results of Section 4.5, that $r$ times the $F$ statistic (4.58) is asymptotically distributed as $\chi^2(r)$.

**4.14** Consider the linear regression model

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{u}, \quad \boldsymbol{u} \sim \mathrm{N}(\mathbf{0}, \sigma^2\mathbf{I}), \quad \mathrm{E}(\boldsymbol{u} \mid \boldsymbol{X}) = \mathbf{0},$$

where $\boldsymbol{X}$ is an $n \times k$ matrix. If $\sigma_0$ denotes the true value of $\sigma$, how is the quantity $\boldsymbol{y}^{\top}\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{y}/\sigma_0^2$ distributed? Use this result to derive a test of the null hypothesis that $\sigma = \sigma_0$. Is this a one-tailed test or a two-tailed test?

**⋆4.15** $P$ values for two-tailed tests based on statistics that have asymmetric distributions are not calculated as in Section 4.2. Let the CDF of the statistic $\tau$ be denoted as $F$, where $F(-x) \neq 1 - F(x)$ for general $x$. Suppose that, for any level $\alpha$, the critical values $c_\alpha^-$ and $c_\alpha^+$ are defined, analogously to (4.05), by the equations

$$F(c_\alpha^-) = \alpha/2 \quad \text{and} \quad F(c_\alpha^+) = 1 - \alpha/2.$$

Show that the marginal significance level associated with a realized statistic $\hat{\tau}$ is $2\min\bigl(F(\hat{\tau}), 1 - F(\hat{\tau})\bigr)$.

**⋆4.16** The rightmost expression in equation (4.62) provides a way to compute the $P$ value for a one-tailed bootstrap test that rejects in the upper tail. Derive comparable expressions for a one-tailed bootstrap test that rejects in the lower tail, for a two-tailed bootstrap test based on a distribution that is symmetric around the origin, and for a two-tailed bootstrap test based on a possibly asymmetric distribution. **Hint:** See Exercise 4.15.