variable z_2, and so the *conditional* distribution of w is normal. Given the conditional mean and variance we have just computed, we see that the conditional distribution must be $\mathrm{N}(b_1z_1, b_2^2)$. The PDF of this distribution is the density of w conditional on z_1, and, by (4.10), it is

$$f(w \mid z_1) = \frac{1}{b_2}\,\phi\Big(\frac{w - b_1z_1}{b_2}\Big). \tag{4.11}$$

In accord with what we noted above, the argument of ϕ here is equal to z_2, which is the standard normal variable corresponding to w conditional on z_1.

The next step is to find the joint density of w and z_1. By (1.15), the density of w conditional on z_1 is the ratio of the joint density of w and z_1 to the marginal density of z_1. This marginal density is just $\phi(z_1)$, since $z_1 \sim \mathrm{N}(0,1)$, and so we see that the joint density is

$$f(w, z_1) = f(z_1)\,f(w \mid z_1) = \phi(z_1)\,\frac{1}{b_2}\,\phi\Big(\frac{w - b_1z_1}{b_2}\Big). \tag{4.12}$$

For the moment, let us suppose that $b_1^2 + b_2^2 = 1$, although we will remove this restriction shortly. Then, if we use (1.06) to get an explicit expression for this joint density, we obtain

$$\begin{aligned} &\frac{1}{2\pi b_2}\exp\Big(-\frac{1}{2b_2^2}\big(b_2^2z_1^2 + w^2 - 2b_1z_1w + b_1^2z_1^2\big)\Big) \\ &\quad = \frac{1}{2\pi b_2}\exp\Big(-\frac{1}{2b_2^2}\big(z_1^2 - 2b_1z_1w + w^2\big)\Big). \end{aligned} \tag{4.13}$$

The right-hand side of equation (4.13) is symmetric with respect to z_1 and w. Thus the joint density can also be expressed as in (4.12), but with z_1 and w interchanged, as follows:

$$f(w, z_1) = \frac{1}{b_2}\,\phi(w)\,\phi\Big(\frac{z_1 - b_1w}{b_2}\Big). \tag{4.14}$$

We are now ready to compute the unconditional, or marginal, density of w. To do so, we integrate the joint density (4.14) with respect to z_1; see (1.12). Note that z_1 occurs only in the last factor on the right-hand side of (4.14). Further, the expression $(1/b_2)\phi\big((z_1 - b_1w)/b_2\big)$, like expression (4.11), is a probability density, and so it integrates to 1. Thus we conclude that the marginal density of w is $f(w) = \phi(w)$, and so it follows that w is standard normal, unconditionally, as we wished to show.

It is now simple to extend this argument to the case for which $b_1^2 + b_2^2 \neq 1$. We define $r^2 = b_1^2 + b_2^2$, and consider w/r. The argument above shows that w/r is standard normal, and so $w \sim \mathrm{N}(0, r^2)$. It is equally simple to extend the result to a linear combination of any number of mutually independent standard normal variables. If we now let w be defined as $b_1z_1 + b_2z_2 + b_3z_3$,