

$E(\eta(x)) \neq \eta(E(x))$. Thus, it is often very easy to calculate plims in circumstances where it would be difficult or impossible to calculate expectations.

However, working with plims can be a little bit tricky. The problem is that many of the stochastic quantities we encounter in econometrics do not have probability limits unless we divide them by n or, perhaps, by some power of n . For example, consider the matrix $\mathbf{X}^\top \mathbf{X}$, which appears in the formula (3.04) for $\hat{\boldsymbol{\beta}}$. Each element of this matrix is a scalar product of two of the columns of \mathbf{X} , that is, two n -vectors. Thus it is a sum of n numbers. As $n \rightarrow \infty$, we would expect that, in most circumstances, such a sum would tend to infinity as well. Therefore, the matrix $\mathbf{X}^\top \mathbf{X}$ does not generally have a plim. However, it is not at all unreasonable to assume that

$$\text{plim}_{n \rightarrow \infty} \frac{1}{n} \mathbf{X}^\top \mathbf{X} = \mathbf{S}_{\mathbf{X}^\top \mathbf{X}}, \quad (3.17)$$

where $\mathbf{S}_{\mathbf{X}^\top \mathbf{X}}$ is a finite nonstochastic matrix with full rank k , because each element of the matrix on the left-hand side of equation (3.17) is now an average of n numbers:

$$\left(\frac{1}{n} \mathbf{X}^\top \mathbf{X} \right)_{ij} = \frac{1}{n} \sum_{t=1}^n x_{ti} x_{tj}.$$

In effect, when we write (3.17), we are implicitly making some assumption sufficient for a LLN to hold for the sequences generated by the squares of the regressors and their cross-products. Thus there should not be too much dependence between $x_{ti} x_{tj}$ and $x_{si} x_{sj}$ for $s \neq t$, and the variances of these quantities should not differ too much as t and s vary.

The OLS Estimator Is Consistent

We can now show that, under plausible assumptions, the least-squares estimator $\hat{\boldsymbol{\beta}}$ is consistent. When the DGP is a special case of the regression model (3.03) that is being estimated, we saw in (3.05) that

$$\hat{\boldsymbol{\beta}} = \boldsymbol{\beta}_0 + (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{u}. \quad (3.18)$$

To demonstrate that $\hat{\boldsymbol{\beta}}$ is consistent, we need to show that the second term on the right-hand side here has a plim of zero. This term is the product of two matrix expressions, $(\mathbf{X}^\top \mathbf{X})^{-1}$ and $\mathbf{X}^\top \mathbf{u}$. Neither $\mathbf{X}^\top \mathbf{X}$ nor $\mathbf{X}^\top \mathbf{u}$ has a probability limit. However, we can divide both of these expressions by n without changing the value of this term, since $n \cdot n^{-1} = 1$. By doing so, we convert them into quantities that, under reasonable assumptions, have nonstochastic plims. Thus the plim of the second term in (3.18) becomes

$$\left(\text{plim}_{n \rightarrow \infty} \frac{1}{n} \mathbf{X}^\top \mathbf{X} \right)^{-1} \text{plim}_{n \rightarrow \infty} \frac{1}{n} \mathbf{X}^\top \mathbf{u} = (\mathbf{S}_{\mathbf{X}^\top \mathbf{X}})^{-1} \text{plim}_{n \rightarrow \infty} \frac{1}{n} \mathbf{X}^\top \mathbf{u} = \mathbf{0}. \quad (3.19)$$