The value $\hat{\kappa}$ that minimizes (12.92) may be found by a non-iterative procedure that is discussed in the appendix. The maximized value of the loglikelihood function is then

$$-\frac{gn}{2}\log 2\pi - \frac{n}{2}\log\hat{\kappa} - \frac{n}{2}\log|\boldsymbol{Y}_*^{\top}\boldsymbol{M_W}\boldsymbol{Y}_*|, \tag{12.93}$$

where $\boldsymbol{Y}_* \equiv [\boldsymbol{y} \quad \boldsymbol{Y}]$.

If we write equation (12.90) as $\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{u}$, then the LIML estimator of $\boldsymbol{\beta}$ is defined by the estimating equations

$$\boldsymbol{X}^{\top}(\mathbf{I} - \hat{\kappa}\boldsymbol{M_W})(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}^{\rm LIML}) = \mathbf{0}, \tag{12.94}$$

which can be solved explicitly once $\hat{\kappa}$ has been computed. We find that

$$\hat{\boldsymbol{\beta}}^{\rm LIML} = \big(\boldsymbol{X}^{\top}(\mathbf{I} - \hat{\kappa}\boldsymbol{M_W})\boldsymbol{X}\big)^{-1}\boldsymbol{X}^{\top}(\mathbf{I} - \hat{\kappa}\boldsymbol{M_W})\boldsymbol{y}. \tag{12.95}$$

A suitable estimate of the covariance matrix of the LIML estimator is

$$\widehat{\rm Var}(\hat{\boldsymbol{\beta}}^{\rm LIML}) = \hat{\sigma}^2\big(\boldsymbol{X}^{\top}(\mathbf{I} - \hat{\kappa}\boldsymbol{M_W})\boldsymbol{X}\big)^{-1}, \tag{12.96}$$

where

$$\hat{\sigma}^2 \equiv \frac{1}{n}(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}^{\rm LIML})^{\top}(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}^{\rm LIML}).$$

Given (12.96), confidence intervals, asymptotic t tests, and Wald tests can readily be computed in the usual way.

Since $\boldsymbol{W} = [\,\boldsymbol{Z} \quad \boldsymbol{W}_1\,]$ is the matrix containing all the instruments, we can decompose $\boldsymbol{M_W}$ as $\boldsymbol{M_Z} - \boldsymbol{P}_{\boldsymbol{M_Z}\boldsymbol{W}_1}$. This makes it clear that $\kappa \geq 1$, since the numerator of (12.92) cannot be smaller than the denominator. If equation (12.90) is just identified, then, by the order condition, $\boldsymbol{Y}$ and $\boldsymbol{W}_1$ have the same number of columns. In this case, it can be shown that the minimized value of κ is actually equal to 1; see Exercise 12.28.

In the context of 2SLS estimation, we saw in Section 8.6 that the Hansen-Sargan test can be used to test overidentifying restrictions. In the case of LIML estimation, it is easier to test these restrictions by a likelihood ratio test. As shown in Exercise 12.28, the maximized loglikelihood of the unconstrained model for which the overidentifying restrictions of (12.90) are relaxed is the same as expression (12.93) for the constrained model, but with $\kappa = 1$. Thus the LR statistic for testing the overidentifying restrictions, which is twice the difference between the unconstrained and constrained maxima, is simply equal to $n\log\hat{\kappa}$. This test statistic was first proposed by Anderson and Rubin (1950). Since there are $l - k$ overidentifying restrictions, the LR statistic is asymptotically distributed as $\chi^2(l - k)$.