3.10 If $\gamma \equiv \boldsymbol{w}^\top \boldsymbol{\beta} = \sum_{i=1}^k w_i \beta_i$, show that $\mathrm{Var}(\hat{\gamma})$, which is given by (3.33), can also be written as

$$\sum_{i=1}^{k} w_i^2 \,\mathrm{Var}(\hat{\beta}_i) + 2\sum_{i=2}^{k}\sum_{j=1}^{i-1} w_i w_j \,\mathrm{Cov}(\hat{\beta}_i, \hat{\beta}_j). \tag{3.68}$$

3.11 Using the data in the file **consumption.data**, construct the variables c_t, the logarithm of consumption, and y_t, the logarithm of income, and their first differences $\Delta c_t = c_t - c_{t-1}$ and $\Delta y_t = y_t - y_{t-1}$. Use these data to estimate the following model for the period 1953:1 to 1996:4:

$$\Delta c_t = \beta_1 + \beta_2 \Delta y_t + \beta_3 \Delta y_{t-1} + \beta_4 \Delta y_{t-2} + \beta_5 \Delta y_{t-3} + \beta_6 \Delta y_{t-4} + u_t. \tag{3.69}$$

Let $\gamma = \sum_{i=2}^6 \beta_i$. Calculate $\hat{\gamma}$ and its standard error in two different ways. One method should explicitly use the result (3.33), and the other should use a transformation of regression (3.69) which allows $\hat{\gamma}$ and its standard error to be read off directly from the regression output.

⋆3.12 Starting from equation (3.42) and using the result proved in Exercise 3.9, but without using (3.43), prove that, if $\mathrm{E}(u_t^2) = \sigma_0^2$ and $\mathrm{E}(u_s u_t) = 0$ for all $s \neq t$, then $\mathrm{Var}(\hat{u}_t) = (1 - h_t)\sigma_0^2$. This is the result (3.44).

3.13 Use the result (3.44) to show that the MM estimator $\hat{\sigma}^2$ of (3.46) is consistent. You may assume that a LLN applies to the average in that equation.

3.14 Prove that $\mathrm{E}(\hat{\boldsymbol{u}}^\top \hat{\boldsymbol{u}}) = (n - k)\sigma_0^2$. This is the result (3.48). The proof should make use of the fact that the trace of a product of matrices is invariant to cyclic permutations; see Section 2.6.

3.15 Consider two linear regressions, one restricted and the other unrestricted:

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{u}, \quad \text{and}$$
$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{Z\gamma} + \boldsymbol{u}.$$

Show that, in the case of mutually orthogonal regressors, with $\boldsymbol{X}^\top \boldsymbol{Z} = \mathbf{O}$, the estimates of $\boldsymbol{\beta}$ from the two regressions are identical.

3.16 Suppose that you use the OLS estimates $\hat{\boldsymbol{\beta}}$, obtained by regressing the $n \times 1$ vector $\boldsymbol{y}$ on the $n \times k$ matrix $\boldsymbol{X}$, to forecast the $n_* \times 1$ vector $\boldsymbol{y}_*$ using the $n_* \times k$ matrix $\boldsymbol{X}_*$. Assuming that the error terms, both within the sample used to estimate the parameters $\boldsymbol{\beta}$ and outside the sample in the forecast period, are $\mathrm{IID}(0, \sigma^2)$, and that the model is correctly specified, what is the covariance matrix of the vector of forecast errors?

3.17 The class of estimators considered by the Gauss-Markov Theorem can be written as $\tilde{\boldsymbol{\beta}} = \boldsymbol{Ay}$, with $\boldsymbol{AX} = \mathbf{I}$. Show that this class of estimators is in fact identical to the class of MM estimators of the form

$$\tilde{\boldsymbol{\beta}} = (\boldsymbol{W}^\top \boldsymbol{X})^{-1} \boldsymbol{W}^\top \boldsymbol{y}, \tag{3.70}$$

where $\boldsymbol{W}$ is a matrix of exogenous variables such that $\boldsymbol{W}^\top \boldsymbol{X}$ is nonsingular.