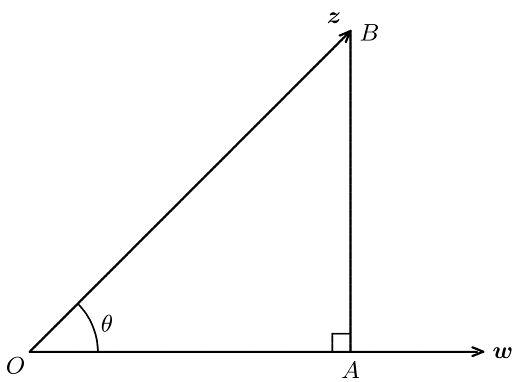

Figure 2.6 The angle between two vectors

the angle BOA is given, by the usual trigonometric rule, by the ratio of the length of the opposite side AB to that of the hypotenuse OB. This ratio is $\sin\theta/1 = \sin\theta$, and so the angle BOA is indeed equal to θ.

Now let us compute the scalar product of $\boldsymbol{w}$ and $\boldsymbol{z}$. It is

$$\langle \boldsymbol{w}, \boldsymbol{z} \rangle = \boldsymbol{w}^\top \boldsymbol{z} = w_1 z_1 + w_2 z_2 = z_1 = \cos\theta,$$

because $w_1 = 1$ and $w_2 = 0$. This result holds for vectors $\boldsymbol{w}$ and $\boldsymbol{z}$ of length 1. More generally, let $\boldsymbol{x} = \alpha\boldsymbol{w}$ and $\boldsymbol{y} = \gamma\boldsymbol{z}$, for positive scalars α and γ. Then $\|\boldsymbol{x}\| = \alpha$ and $\|\boldsymbol{y}\| = \gamma$. Thus we have

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^\top \boldsymbol{y} = \alpha\gamma\boldsymbol{w}^\top \boldsymbol{z} = \alpha\gamma\langle \boldsymbol{w}, \boldsymbol{z} \rangle.$$

Because $\boldsymbol{x}$ is parallel to $\boldsymbol{w}$, and $\boldsymbol{y}$ is parallel to $\boldsymbol{z}$, the angle between $\boldsymbol{x}$ and $\boldsymbol{y}$ is the same as that between $\boldsymbol{w}$ and $\boldsymbol{z}$, namely θ. Therefore,

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \|\boldsymbol{x}\|\,\|\boldsymbol{y}\|\cos\theta. \tag{2.07}$$

This is the general expression, in geometrical terms, for the scalar product of two vectors. It is true in E^n just as it is in E^2, although we have not proved this. In fact, we have not quite proved (2.07) even for the two-dimensional case, because we made the simplifying assumption that the direction of $\boldsymbol{x}$ and $\boldsymbol{w}$ is horizontal. In Exercise 2.1, we ask the reader to provide a more complete proof.

The cosine of the angle between two vectors provides a natural way to measure how close two vectors are in terms of their directions. Recall that $\cos\theta$ varies between -1 and 1; if we measure angles in radians, $\cos 0 = 1$, $\cos\pi/2 = 0$, and $\cos\pi = -1$. Thus $\cos\theta$ is 1 for vectors that are parallel, 0 for vectors that are at right angles to each other, and -1 for vectors that point in directly