When the gradient vector of the unrestricted loglikelihood function is evaluated at the restricted estimates $\tilde{\boldsymbol{\theta}}$, the first $k - r$ elements, which are the elements of the vector $\boldsymbol{g}_1(\tilde{\boldsymbol{\theta}})$, are zero, by equation (10.63). However, the r–vector $\boldsymbol{g}_2(\tilde{\boldsymbol{\theta}})$, which contains the remaining r elements, is in general nonzero. In fact, a Taylor expansion gives

$$n^{-1/2}\boldsymbol{g}_2(\tilde{\boldsymbol{\theta}}) = n^{-1/2}\boldsymbol{g}_2(\boldsymbol{\theta}_0) + n^{-1}\boldsymbol{H}_{21}(\bar{\boldsymbol{\theta}})\, n^{1/2}(\tilde{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_1^0), \tag{10.65}$$

where our usual shorthand notation $\bar{\boldsymbol{\theta}}$ is used for a vector that tends to $\boldsymbol{\theta}_0$ as $n \to \infty$, and $\boldsymbol{H}_{21}(\cdot)$ is the lower left block of the Hessian of the loglikelihood. The information matrix equality (10.34) shows that the limit of (10.65) for a correctly specified model is

$$\begin{aligned}
\operatorname*{plim}_{n\to\infty} n^{-1/2}\boldsymbol{g}_2(\tilde{\boldsymbol{\theta}}) &= \operatorname*{plim}_{n\to\infty} n^{-1/2}\boldsymbol{g}_2(\boldsymbol{\theta}_0) - \mathcal{I}_{21}^0 \operatorname*{plim}_{n\to\infty} n^{1/2}(\tilde{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}_1^0) \\
&= \operatorname*{plim}_{n\to\infty}\big(n^{-1/2}\boldsymbol{g}_2(\boldsymbol{\theta}_0) - \mathcal{I}_{21}^0(\mathcal{I}_{11}^0)^{-1}n^{-1/2}\boldsymbol{g}_1(\boldsymbol{\theta}_0)\big) \\
&= [\,-\mathcal{I}_{21}^0(\mathcal{I}_{11}^0)^{-1} \quad \mathbf{I}\,] \operatorname*{plim}_{n\to\infty} \begin{bmatrix} n^{-1/2}\boldsymbol{g}_1(\boldsymbol{\theta}_0) \\ n^{-1/2}\boldsymbol{g}_2(\boldsymbol{\theta}_0) \end{bmatrix},
\end{aligned} \tag{10.66}$$

where $\mathcal{I}^0 \equiv \mathcal{I}(\boldsymbol{\theta}_0)$, $\mathbf{I}$ is an $r \times r$ identity matrix, and the second line follows from (10.64).

Since the variance of the full gradient vector, $\operatorname{plim} n^{-1/2}\boldsymbol{g}(\boldsymbol{\theta})$, is just $\mathcal{I}_0$, the variance of the last expression in (10.66) is

$$\begin{aligned}
\operatorname{Var}\big(\operatorname*{plim}_{n\to\infty} n^{-1/2}\boldsymbol{g}_2(\tilde{\boldsymbol{\theta}})\big) &= [\,-\mathcal{I}_{21}^0(\mathcal{I}_{11}^0)^{-1} \quad \mathbf{I}\,] \begin{bmatrix} \mathcal{I}_{11}^0 & \mathcal{I}_{12}^0 \\ \mathcal{I}_{21}^0 & \mathcal{I}_{22}^0 \end{bmatrix} \begin{bmatrix} -(\mathcal{I}_{11}^0)^{-1}\mathcal{I}_{12}^0 \\ \mathbf{I} \end{bmatrix} \\
&= \mathcal{I}_{22}^0 - \mathcal{I}_{21}^0(\mathcal{I}_{11}^0)^{-1}\mathcal{I}_{12}^0.
\end{aligned} \tag{10.67}$$

In Exercise 7.11, expressions were developed for the blocks of the inverses of partitioned matrices. It is easy to see from those expressions that the inverse of (10.67) is the 22 block of $\mathcal{I}^{-1}(\boldsymbol{\theta}_0)$. Thus, in order to obtain a statistic in asymptotically χ^2 form based on $\boldsymbol{g}_2(\tilde{\boldsymbol{\theta}})$, we can construct the quadratic form

$$\mathrm{LM} = n^{-1/2}\boldsymbol{g}_2^{\top}(\tilde{\boldsymbol{\theta}})(\tilde{\mathcal{I}}^{-1})_{22}\, n^{-1/2}\boldsymbol{g}_2(\tilde{\boldsymbol{\theta}}) = \boldsymbol{g}_2^{\top}(\tilde{\boldsymbol{\theta}})(\tilde{\boldsymbol{I}}^{-1})_{22}\, \boldsymbol{g}_2(\tilde{\boldsymbol{\theta}}), \tag{10.68}$$

in which $\tilde{\mathcal{I}} = n^{-1}\boldsymbol{I}(\tilde{\boldsymbol{\theta}})$, and the notations $(\tilde{\mathcal{I}}^{-1})_{22}$ and $(\tilde{\boldsymbol{I}}^{-1})_{22}$ signify the 22 blocks of the inverses of $\tilde{\mathcal{I}}$ and $\boldsymbol{I}(\tilde{\boldsymbol{\theta}})$, respectively.

Since the statistic (10.68) is a quadratic form in an r–vector, which is asymptotically normally distributed with mean $\mathbf{0}$, and the inverse of an $r \times r$ matrix that consistently estimates the covariance matrix of that vector, it is clear that the LM statistic is asymptotically distributed as $\chi^2(r)$ under the null. However, expression (10.68) is notationally awkward. Because the first-order